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LETTER TO THE EDITOR

Classical dynamical origin of Feynman paths?

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Abstract. Even though only the classical action explicitly appears in the Feynman path integral, quantum fluctuations show up in the fluctuating nature of Feynman paths. This prevents a *classical interpretation* of the latter objects. Still, it makes sense to ask whether the Feynman paths have any relation to the classical dynamical trajectories in configuration space—a possibility naturally suggested by the semiclassical approximation. Unfortunately, no such relation exists (in general). Things are quite different for the generalized Feynman paths, which actually contribute in a novel path integral giving an alternative representation of the quantum mechanical propagator. We show that these paths fluctuate about a *classical dynamical trajectory* (in configuration space) because of a certain white noise. That is to say, the generalized Feynman paths are fluctuating curves that collapse on a classical dynamical trajectory as $\hbar \to 0$. This result throws new light on the interplay between classical and quantum mechanics and hints at a new approach to the quantum theory.

The Feynman path integral plays nowadays a central role in quantum physics and offers a more intuitive view of quantization as compared to the canonical operator approach.

Our aim is to discuss certain features of this strategy which seem to have attracted little attention so far. This will be done by considering non-relativistic quantum mechanics corresponding to the classical Lagrangian

$$L(x, \dot{x}, t) = \tfrac{1}{2} m \dot{x}_i \dot{x}_i + \Omega_i(x, t) \dot{x}_i - \Phi(x, t) \tag{1}$$

which describes a point particle $\mathcal{S}$ (mass m, no spin) with configuration space $\mathcal{M} = \mathfrak{R}^N$. Some generalizations will be presented elsewhere.

As is well known, the Feynman path integral representation of the propagator reads (Feynman and Hibbs 1965, Kleinert 1990, Schulman 1981)

$$\langle x'', t'' | x', t' \rangle = \int \mathfrak{D}x(t) \delta(x'' - x(t'')) \delta(x' - x(t')) \exp\{(i/\hbar) S[x(\cdot)]_{t'}^{t''}\} \tag{2}$$

with $S[x(\cdot)]_{t'}^{t''}$ denoting the classical action along a generic path $x(t) \in \mathcal{M}$

$$S[x(\cdot)]_{t'}^{t''} \equiv \int_{t'}^{t''} \mathrm{d}t \, L(x(t), \dot{x}(t), t). \tag{3}$$

The paths which actually contribute in the path integral (2) are commonly referred to as *Feynman paths*. Although all (continuous) trajectories joining (x', t') with (x'', t'') seem to enter the Feynman path integral, it turns out that the Feynman paths form a certain subset of fractal curves with Hausdorff dimension *two*, namely they are

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characterized by the property $\Delta x(t)\sim(\Delta t)^{1/2}$. This fact entails that these objects have a fluctuating nature quite similar to that of the erratic trajectories familiar from the theory of macroscopic Brownian motion. All this should not come as a surprise after all, since the Feynman approach to quantum mechanics is structurally very similar to the Wiener–Onsager–Machlup formulation of classical stochastic diffusion processes in $\mathcal{M}=\mathfrak{R}^N$ (within this analogy, classical probabilities are obviously replaced by complex quantum amplitudes whereas the diffusion constant D is formally imaginary in quantum mechanics and is proportional to $\hbar$). As a consequence, the Feynman paths are conceptually on the same footing as the *Wiener paths*† which enjoy the property $\Delta x(t)\sim(\Delta t)^{1/2}$ as well.

Actually, the nature of Wiener paths is best clarified by the Langevin description of classical stochastic diffusion processes. Since this circumstance is very important for the considerations that we have in mind, we shall briefly discuss this point before addressing the question asked in the title. Consider a diffusion process in $\mathcal{M}=\mathfrak{R}^N$ with diffusion constant D and drift velocity $V(x,t)$ (Gardiner 1985, Risken 1984, Van Kampen 1981). Then the sample paths of the process in question are given by the *Langevin equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_i(t)=V_i(\xi(t),t)+(2D)^{1/2}\eta_i(t) \tag{4}$$

where $\eta(t)\equiv\{\eta_i(t)\}_{1\leqslant i\leqslant N}$ are *Gaussian white noise* variables defined by the functional (probability) measure

$$\mathscr{D}\mu[\eta(\cdot)]\sim\mathscr{D}\eta(t)\exp\left\{-(1/2)\int_{-\infty}^{+\infty}\mathrm{d}t\,\eta_i(t)\eta_i(t)\right\}. \tag{5}$$

We denote by $\xi(t;x',t';[\eta(\cdot)])$ the solution to (4) with initial condition $\xi(t')=x'$. Because of (5), it follows that these solutions are fractal curves with Hausdorff dimension two, that is to say $\Delta\xi(t;x',t';[\eta(\cdot)])\sim(\Delta t)^{1/2}$. Moreover, whenever fluctuations can be neglected, equation (4) reduces to the deterministic equation

$$\frac{\mathrm{d}}{\mathrm{d}t}q_i(t)=V_i(q(t),t) \tag{6}$$

whose solution with initial condition $q(t')=x'$ will be denoted by $q(t;x',t')$. Correspondingly, as $D\to 0$ we have $\xi(t;x',t';[\eta(\cdot)])\to q(t;x',t')$ for any noise sample $\eta(t)$. All this means that the sample paths $\xi(t;x',t';[\eta(\cdot)])$ can be viewed as fluctuating curves about the deterministic trajectory $q(t;x',t')$. Now, it can be shown that the Wiener paths have the same (functional) probability distribution as the solutions of (4), and so the Wiener paths joining (x',t') with (x'',t'') can be understood as trajectories $\xi(t;x',t';[\eta(\cdot)])$ which fulfil the condition

$$\xi(t'';x',t';[\eta(\cdot)])=x'' \tag{7}$$

for all Gaussian white noise configurations $\eta(t)$. Therefore we see that the considered Wiener paths fluctuate about the deterministic trajectory $q(t;x',t')$ given by (6) because of a Gaussian white noise‡.

† They are analogously defined as the paths which contribute in the Wiener–Onsager–Machlup functional integral.

‡ All this is of course well known to people working in the field of stochastic processes. However, as far as we can see, several physicists interested in quantum mechanics seem not to be quite familiar with stochastic processes. We have therefore decided to briefly review these matters here.

Coming back to quantum mechanics, one naturally wonders whether a similar mechanism underlies the Feynman path integral. Can the Feynman paths be regarded as solutions of a certain Langevin equation? Suppose momentarily that the answer lies in the positive. Then the Feynman paths could be viewed—in the same sense as before—as fluctuating curves about a deterministic trajectory (which obeys the Langevin equation in question with the noise term discarded). What is the physical meaning of such a deterministic trajectory?

An important virtue of the Feynman formulation is to provide an appealing connection between classical and quantum dynamics, in that the classical action appears in (2). We stress that quantum fluctuations show up within the Feynman path integral precisely in the fluctuating nature of Feynman paths, making them consistent with the uncertainty relations (Feynman and Hibbs 1965). Hence these paths do not possess any classical meaning. We also mention that amplitudes—but not probabilities obeying classical Kolmogorov axioms—can consistently be associated with the Feynman paths, so that they cannot be interpreted as trajectories followed by S in the classical sense. Consider now the behaviour of the Feynman path integral in the asymptotic limit $\hbar \to 0$. Manifestly, quantum fluctuations tend to disappear and concomitantly classical dynamics emerges via a stationary action mechanism. Thus, we are led to expect that there should be some relation between the Feynman paths and the classical dynamical trajectories. Can the Feynman paths be actually recognized as fluctuating curves about a classical dynamical trajectory?

Remarkably enough, the whole issue can be settled in a fairly simple way. We start by observing that the path integral (2) with Lagrangian (1) can be cast in the form

$$\langle x'', t''|x', t'\rangle = \int \mathscr{D}x(t)\delta(x''-x(t''))\delta(x'-x(t'))$$

$$\times \exp\left\{(\mathrm{i}/\hbar)(m/2)\int_{t'}^{t''} \mathrm{d}t\left[\dot{x}_i(t)+\frac{1}{m}\Omega_i(x(t),t)\right]^2\right\}$$

$$\times \exp\left\{-(\mathrm{i}/\hbar)\int_{t'}^{t''}\mathrm{d}t\left[\frac{1}{2m}\Omega_i(x(t),t)\Omega_i(x(t),t)+\Phi(x(t),t)\right]\right\}. \tag{8}$$

Further, the first exponential in (8) can be rewritten as†

$$\exp\left\{(\mathrm{i}/\hbar)(m/2)\int_{t'}^{t''}\mathrm{d}t\left[\dot{x}_i(t)+\frac{1}{m}\Omega_i(x(t),t)\right]^2\right\}$$

$$\sim \int \mathscr{D}\mu[\eta(\cdot)]\delta\left[\dot{x}(t)+\frac{1}{m}\Omega(x(t),t)-\left(\frac{\hbar}{m}\right)^{1/2}\eta(t)\right]_{t'}^{t''} \tag{9}$$

with the definition

$$\mathscr{D}\mu[\eta(\cdot)] \sim \mathscr{D}\eta(t)\exp\left\{(\mathrm{i}/2)\int_{-\infty}^{+\infty}\mathrm{d}t\,\eta_i(t)\eta_i(t)\right\}. \tag{10}$$

Observe that (10) is very similar to (5), and so $\eta(t) \equiv \{\eta_i(t)\}_{1 \leqslant i \leqslant N}$ will be naturally referred to as *Fresnel white noise* variables (no confusion will arise, for only the Fresnel white noise will be considered from now on). Physically, equation (10) should

† The notation $\delta[f(t)]_{t'}^{t''}$ means a functional delta function, namely the continuous product of ordinary delta functions $\delta(f(t))$ for all values of t between t' and t''.

be interpreted as an amplitude (pseudo) measure, in agreement with the fact that probabilities get replaced by amplitudes in quantum mechanics. Next, we make use of the well known identity (Zinn-Justin 1989)

$$\delta[x(t)-\xi(t;x',t';[\eta(\cdot)])]_{t'}^{t''}=\delta\left[\dot{x}(t)+\frac{1}{m}\Omega(x(t),t)-\left(\frac{\hbar}{m}\right)^{1/2}\eta(t)\right]_{t'}^{t''} \times\delta(x'-x(t'))J[x(\cdot)] \tag{11}$$

where $J[x(\cdot)]$ is the functional Jacobian $\det|\delta[\dot{x}_i(t)+\Omega_i(x(t),t)/m]/\delta x_j(t')|$ and $\xi(t;x',t';[\eta(\cdot)])$ denotes the solution (with initial condition $\xi(t')=x'$) to the following Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_i(t)=-\frac{1}{m}\Omega_i(\xi(t),t)+\left(\frac{\hbar}{m}\right)^{1/2}\eta_i(t). \tag{12}$$

Since $\mathcal{D}\mu[\eta(\cdot)]$ in (10) defines an amplitude distribution, probabilities obeying classical Kolmogorov axioms cannot be associated with the solutions of (12), whose interpretation is therefore the same as for Feynman paths. Quite similarly to what happens for (4), we have $\Delta\xi(t;x',t';[\eta(\cdot)])\sim(\Delta t)^{1/2}$ as a consequence of (10). Moreover, in the limit $\hbar\to0$, equation (12) manifestly reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{q}_i(t)=-\frac{1}{m}\Omega_i(\bar{q}(t),t). \tag{13}$$

We denote by $\bar{q}(t;x',t')$ the solution to (13) (with initial condition $\bar{q}(t')=x'$). Thus, as $\hbar\to0$ we have $\xi(t;x',t';[\eta(\cdot)])\to\bar{q}(t;x',t')$ for any noise sample $\eta(t)$. Again, all this means that the solutions $\xi(t;x',t';[\eta(\cdot)])$ can be viewed as fluctuating curves about $\bar{q}(t;x',t')$. Combining now (9) and (11) together, we get

$$\begin{aligned}\int &\mathcal{D}x(t)\delta(x''-x(t''))\delta(x'-x(t'))\exp\left\{(\mathrm{i}/\hbar)(m/2)\int_{t'}^{t''}\mathrm{d}t\left[\dot{x}_i(t)+\frac{1}{m}\Omega_i(x(t),t)\right]^2\right\}\\ &\times\exp\left\{-(\mathrm{i}/\hbar)\int_{t'}^{t''}\mathrm{d}t\left[\frac{1}{2m}\Omega_i(x(t),t)\Omega_i(x(t),t)+\Phi(x(t),t)\right]\right\}\\ &\sim\int\mathcal{D}x(t)\int\mathcal{D}\mu[\eta(\cdot)]\,\delta(x''-\xi(t'';x',t';[\eta(\cdot)]))\\ &\times\delta[x(t)-\xi(t;x',t';[\eta(\cdot)])]_{t'}^{t''}\\ &\times J[x(\cdot)]^{-1}\exp\left\{-(\mathrm{i}/\hbar)\int_{t'}^{t''}\mathrm{d}t\left[\frac{1}{2m}\Omega_i(x(t),t)\Omega_i(x(t),t)+\Phi(x(t),t)\right]\right\}\end{aligned} \tag{14}$$

showing that the Feynman paths joining (x',t') with (x'',t'') can be understood as trajectories $\xi(t;x',t';[\eta(\cdot)])$ which fulfil the condition

$$\xi(t'';x',t';[\eta(\cdot)])=x'' \tag{15}$$

for all Fresnel white noise configurations $\eta(t)$. We conclude that the Feynman paths in question fluctuate about the deterministic trajectory $\bar{q}(t;x',t')$ given by (13) because of the Fresnel white noise. Unfortunately, $\bar{q}(t;x',t')$ is not (in general) a classical dynamical trajectory of $\mathcal{S}$ in $\mathcal{M}$ (indeed, by taking the time derivative of (13) it is straightforward to see that $\bar{q}(t;x',t')$ does not obey the Lagrange equations corresponding to Lagrangian (1) for an arbitrary choice of the potentials). Thus, the question asked in the title has a negative answer.

Yet, the situation changes drastically when the attention is turned to the *new path integral* which enters the alternative representation of the propagator (Roncadelli 1992)

$$\langle x'',t''|x',t'\rangle = \exp\{(\mathrm{i}/\hbar)[S(x'',t'')-S(x',t')]\}\int \mathscr{D}x(t)\,\delta(x''-x(t''))\delta(x'-x(t'))$$
$$\times\exp\left\{(\mathrm{i}/\hbar)(m/2)\int_{t'}^{t''}\mathrm{d}t[\dot{x}_i(t)-\mathscr{V}_i(x(t),t;[S(\cdot)])]^2\right\} \tag{16}$$

where we have set

$$\mathscr{V}_i(x,t;[S(\cdot)])\equiv\frac{1}{m}\left(\frac{\partial}{\partial x_i}S(x,t)-\Omega_i(x,t)\right) \tag{17}$$

and $S(x,t)$ denotes throughout an arbitrary solution of the classical Hamilton–Jacobi equation associated with Lagrangian (1)

$$\frac{\partial}{\partial t}S(x,t)+\frac{1}{2m}\left(\frac{\partial}{\partial x_i}S(x,t)-\Omega_i(x,t)\right)^2+\Phi(x,t)=0. \tag{18}$$

The paths which actually contribute in the path integral (16) for a given $S(x,t)$ will be referred to as generalized Feynman paths controlled by $S(x,t)$ (since the integrand depends on $S(x,t)$ so, in general, will the paths under consideration). Also the generalized Feynman paths are fractal curves with Hausdorff dimension two (namely for which $\Delta x(t)\sim(\Delta t)^{1/2}$) and there are infinitely-many equivalent sets of generalized Feynman paths—each controlled by a particular solution $S(x,t)$—for the RHS of (16) does not (globally) depend on which specific $S(x,t)$ is used†. Before proceeding further, we recall that once a particular (arbitrary) integral $S(x,t)$ is known, a family of trajectories in $\mathscr{M}$ is provided by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}q_i(t)=\mathscr{V}_i(q(t),t;[S(\cdot)]). \tag{19}$$

If we denote by $q(t;x',t';[S(\cdot)])$ the solution to (19) with initial condition $q(t')=x'$ and controlled by $S(x,t)$, then $q(t;x',t';[S(\cdot)])$ is just the classical dynamical trajectory of $\mathscr{S}$ in $\mathscr{M}$ selected by the initial data $q(t')=x'$, $p(t')=(\nabla S)(x',t')$ (Arnold 1978).

Now, the very strong structural similarity between (19) and the exponent in the path integral (16) suggests that the generalized Feynman paths might have a simple relation to the classical dynamical trajectories in $\mathscr{M}$. As we shall demonstrate below, this is indeed the case and the generalized Feynman paths possess a *classical dynamical origin*.

† All previous remarks about the interpretation of Feynman paths apply to generalized Feynman paths as well.

Let us apply to the new path integral the same formal manipulations which led us to (14) starting from the Feynman path integral. We find

$$\int \mathcal{D}x(t)\delta(x''-x(t''))\delta(x'-x(t'))\exp\left\{\frac{im}{2\hbar}\int_{t'}^{t''}\mathrm{d}t[\dot{x}_i(t)-\mathcal{V}_i(x(t),t;[s(\cdot)])]^2\right\}$$

$$\sim\int \mathcal{D}x(t)\int \mathcal{D}\mu[\eta(\cdot)]\delta(x''-\xi(t'';x',t';[S(\cdot),\eta(\cdot)]))$$

$$\times\,\delta[x(t)-\xi(t;x',t';[S(\cdot),\eta(\cdot)])]_{t'}^{t''}J[x(\cdot)]^{-1} \tag{20}$$

where $\xi(t;x',t';[S(\cdot),\eta(\cdot)])$ denotes the solution (with initial contdition $\xi(t')=x'$ and controlled by $S(x,t)$) to the following Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi_i(t)=\mathcal{V}_i(\xi(t),t;[S(\cdot)])+\left(\frac{\hbar}{m}\right)^{1/2}\eta_i(t) \tag{21}$$

(the meaning of all other symbols is the same as for the previous case). Then (20) shows that the generalized Feynman paths controlled by $S(x,t)$ which join (x',t') with (x'',t'') can be recognized as trajectories $\xi(t;x',t';[S(\cdot),\eta(\cdot)])$ fulfilling the condition

$$\xi(t'';x';t';[S(\cdot),\eta(\cdot)])=x'' \tag{22}$$

for all Fresnel white noise configurations $\eta(t)$. Now, in the limit $\hbar\to 0$, (21) manifestly reduces to (19), and so as $\hbar\to 0$ we have $\xi(t;x',t';[S(\cdot),\eta(\cdot)])\to q(t;x',t';[S(\cdot)])$ for any noise sample $\eta(t)$. Once again, all this entails that the solutions $\xi(t;x',t';[S(\cdot),\eta(\cdot)])$ are fluctuating curves about the classical dynamical trajectory $q(t;x',t';[S(\cdot)])$ controlled by the same $S(x,t)$†.

We conclude that the generalized Feynman paths (entering the path integral (16)) controlled by $S(x,t)$ fluctuate about the classical dynamical trajectory in $\mathcal{M}$ defined by initial data $q(t')=x', p(t')=(\nabla S)(x',t')$. More generally, we see that the generalized Feynman paths in question arise from the classical dynamical trajectory $q(t;x',t';[S(\cdot)])$ as a consequence of the Fresnel white noise.

In our opinion, the new path integral (16) is much more closely related to classical dynamics than the Feynman path integral. Not only are all quantities which explicitly appear in it genuinely classical, but also the integration paths possess a classical dynamical origin. Accordingly, (16) can be interpreted in a very suggestive pictorial way (which would be impossible for the Feynman path integral). Choose an arbitrary solution $S(x,t)$ of (18) and consider the corresponding classical dynamical trajectory $q(t;x',t';[S(\cdot)])$. Then the quantum mechanical propagator $\langle x'',t''|x',t'\rangle$ arises—up to the classical exponential prefactor—by summing the classical object

$$\exp\left\{(\mathrm{i}/\hbar)(m/2)\int_{t'}^{t''}\mathrm{d}t[\dot{x}_i(t)-\mathcal{V}_i(x(t),t;[S(\cdot)])]^2\right\} \tag{23}$$

† A comment about this result is perhaps in order. Since $S(x,t)$ is an arbitary solution to (18), it can well happen that $q(t'';x',t';[S(\cdot)])$ is very far away from x''. Still, the generalized Feynman paths controlled by the same $S(x,t)$ should go through x'' (this follows from their very definition). But then one can well wonder whether fluctuations are able to make $\xi(t'';x',t';[S(\cdot),\eta(\cdot)])$ coincide with x''. As a matter of fact, exactly the same situation occurs in the Langevin description of a classical diffusion process. Indeed, since the drift in (6) is obviously independent of x'', it often happens that $q(t'';x',t')$ is largely different from x'', and yet the Wiener paths go through x''. So, one is again led to wonder how can fluctuations achieve $\xi(t'';x',t';[\eta(\cdot)])=x''$. All we can tell is that mathematically the mechanism works, in spite of the fact that we have not been able to find a more intuitive explanation.

over all trajectories that fluctuate about $q(t; x', t'; [S(\cdot)])$ because of the Fresnel white noise and happen to end up in x'' at time t''. Different solutions $S(x, t)$ would give rise to different classical dynamical trajectories in $\mathcal{M}$. Still, they all lead to the same quantum mechanical time evolution! This explains why the dependence on the initial momentum gets washed out on going over to the quantum theory (recall that $p(t') = (\nabla S)(x', t')$). Besides interesting in its own right, this scenario suggests quite naturally that quantum mechanics could be reformulated by taking (18), (21) and (10) as a starting point. The Fresnel white noise would then bring quantum fluctuations into the otherwise classical game (these matters will be discussed elsewhere).

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